MINIMAL MUTATION-INFINITE QUIVERS

JOHN LAWSON

Abstract. Quivers constructed from hyperbolic Coxeter simplices give examples of minimal mutation-infinite quivers, however they are not the only such quivers. We classify minimal mutation-infinite quivers through a number of moves and link the representatives of the classes with the hyperbolic Coxeter simplices, plus exceptional classes which are not related to simplices.

1. Introduction

Mutations on quivers were introduced by Fomin and Zelevinsky in their introduction to cluster algebras in 2002 [5]. Since then this area has been widely studied, with applications in numerous areas of mathematics.

The mutation class of a quiver is the collection of all quivers which can be obtained from the original through a sequence of mutations. All finite sized mutation classes were classified by Felikson, Shapiro and Tumarkin in [4], this classification necessarily contains the mutation-classes of quivers that give finite-type cluster algebras, which were classified by Fomin and Zelevinsky in [6]. This classification states that all finite-type cluster algebras come from quivers given by orientations of Dynkin diagrams, while quivers from orientations of affine Dynkin diagrams are also mutation-finite.

In their classification Fomin and Zelevinsky introduced mutations on diagrams, and Seven classified all minimal 2-infinite diagrams in [9]. The work by Seven on minimal 2-infinite diagrams inspired the study of minimal mutation-infinite quivers and this paper builds on work done by Felikson, Shapiro and Tumarkin in [4, Section 7] proving a number of useful results about minimal mutation-infinite quivers.

Minimal mutation-infinite quivers are those which belong to an infinite mutation class, but any subquiver belongs to a finite mutation class. Simply-laced diagrams from hyperbolic Coxeter simplices of finite volume have the property that any subdiagram is a Dynkin or affine Dynkin diagram and so any mutation-infinite orientation of such a diagram is minimal mutation-infinite. The motivating question behind this study is whether the family of minimal mutation-infinite quivers from orientations of hyperbolic Coxeter simplex diagrams contains all minimal mutation-infinite quivers.

In this paper we classify all minimal mutation-infinite quivers, with classes represented by orientations of hyperbolic Coxeter simplex diagrams as well as some exceptional representatives. The classification is defined in terms of moves, which are specific sequences of mutations. In general, mutation does not preserve the property of a quiver being minimal mutation-infinite, however the moves are constructed in such a way that they do.

Theorem 5.1. *Any minimal mutation-infinite quiver with at most* 9 *vertices can be transformed through sink-source mutations and at most 5 moves to one of an orientation of a hyperbolic Coxeter diagram, a double arrow quiver or an exceptional quiver.*

Theorem 5.2. *Any minimal mutation-infinite quiver can be transformed through sink-source mutations and at most 10 moves to one of an orientation of a hyperbolic Coxeter diagram, a double arrow quiver or an exceptional quiver.*

The results of this paper give a procedure to check whether any given quiver is mutation-infinite without having to compute any part of its mutation class. This procedure follows from the fact that any mutation-infinite quiver must contain a minimal mutation-infinite complete subquiver.

In Section 2 of this paper we remind the reader of the process of mutating quivers, and recall the properties arising from mutation-equivalence of quivers. Using these definitions we introduce

The author's PhD studies are supported by an EPSRC studentship.

minimal mutation-infinite quivers and highlight the interest behind their study. In Section 3 we recall the relations between quivers, diagrams and Coxeter simplices, as well as constructing quivers from orientations of certain Coxeter diagrams given by these simplices. Some examples of these quivers give minimal mutation-infinite quivers.

Section 5 introduces a classification of all minimal mutation-infinite quivers through a number of elementary moves defined in Section 4 and listed in Appendix B. These moves allow minimal mutation-infinite quivers to be transformed to other minimal mutation-infinite quivers and so admit a classification of such quivers.

The quiver classification involved a large computational effort to find all minimal mutation-infinite quivers. Appendix A details the procedures used in this computation. Details about implementations of these procedures and the complete lists of minimal mutation-infinite quivers can be found on the author's website [8].

Acknowledgements

The author would like to thank John Parker and Pavel Tumarkin for their supervision and support.

2. Mutations

The following gives an introduction to mutations of quivers. Further information and the extension to cluster algebras can be found in introductory survey articles such as [12] or [7]. Given a graph denote a cycle of length one as a loop, a cycle of length two as a 2-cycle.

Definition 2.1. A **quiver** is an oriented graph with possibly more than one arrow between any two vertices. In the following a quiver is always considered with the additional restriction that it contains no loops or 2-cycles.

This restriction ensures that the quiver is uniquely determined by its skew-symmetric adjacency (or exchange) matrix. This correspondence depends on an indexing of the vertices of the quiver and it is convenient to always consider the quiver with such a numbering, so that any vertex can be referred to by its index.

Definition 2.2. **Mutation** of a quiver is a function at a vertex k of the quiver which changes the arrows around the vertex according to 3 rules:

(1) Whenever there is a path through vertex k of the form $i \to k \to j$ then add an arrow $i \to j$.
(2) Reverse the direction of all arrows adjacent to k
(3) Remove a maximal collection of 2-cycles created in this process.

Figure 1 shows an example of mutation on a quiver.

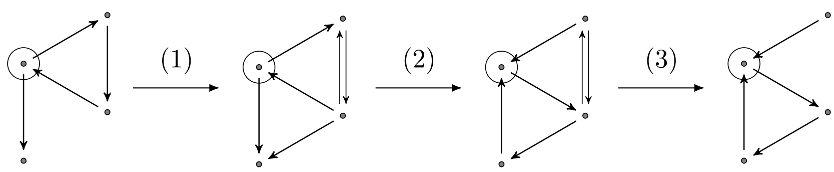


Figure 1. Example of mutation at the circled vertex. Each step in Definition 2.2 is shown separately.

Definition 2.3. Two quivers P and Q are **mutation-equivalent** if there exists a sequence of mutations taking P to Q. The **mutation-class** of a quiver is the equivalence class under this equivalence relation. A quiver is **mutation-finite** if it belongs to a mutation-class of finite size, otherwise the quiver is **mutation-infinite**.

All mutation-finite quivers have been classified by Felikson, Shapiro and Tumarkin in their paper [4] as either a quiver arising from an orientation of a triangulation of a surface or a quiver in one of 11 exceptional mutation-classes.

2.1. **Partial ordering on quivers.** A partial ordering can be put on all quivers given by inclusion of complete subquivers.

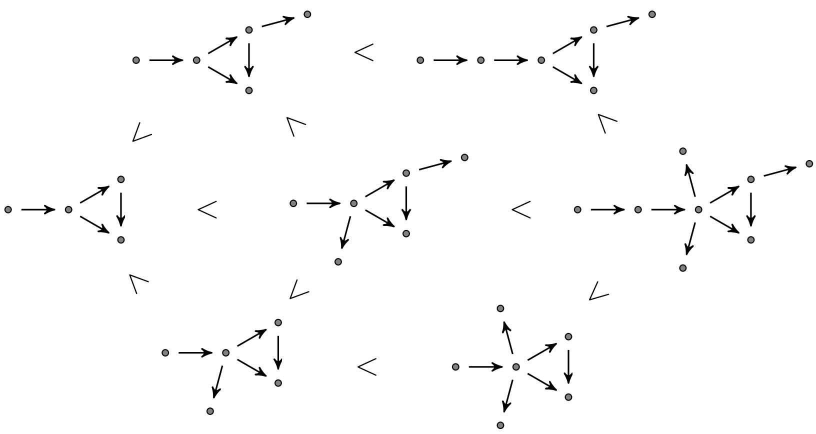


FIGURE 2. The partial ordering on some examples of quivers.

Definition 2.4. Given two quivers P and Q, then $P < Q$ if P can be obtained by removing vertices (and all arrows adjacent to each removed vertex) from Q. Equivalently, if B_P and B_Q are the exchange matrices of P and Q respectively, then $P < Q$ if B_P is a submatrix of B_Q up to simultaneously permuting the rows and columns of B_P. If $P < Q$ then call P a **complete subquiver** of Q.

For brevity it is convenient to omit the word complete and write subquiver to mean complete subquiver. Denote the vertices of Q as $u_1, \ldots, u_m, v_1, \ldots, v_n$ and let P be the subquiver of Q obtained by removing vertices $v_1, \ldots, v_n$. Then any mutation at u_i commutes with removing these vertices $\{v_j\}$ shown in Figure 3, giving the following proposition.

$$\begin{tikzcd} Q \arrow[r, "\text{remove } \{v_j\}"] \arrow[d] & P \arrow[d] \\ \mu_{u_i}(Q) \arrow[r, "\text{remove } \{v_j\}"'] & \mu_{u_i}(P) \end{tikzcd}$$

FIGURE 3. Commutative diagram showing mutations and vertex removal.

Proposition 2.5. *A quiver which contains some mutation-infinite quiver as a subquiver is necessarily mutation-infinite. Equivalently any subquiver of a mutation-finite quiver is mutation-finite.*

Proposition 2.5 shows that there are minimal mutation-infinite quivers with respect to the above partial ordering. Equivalently these minimal mutation-infinite quivers could be defined as follows:

Definition 2.6. A **minimal mutation-infinite quiver** is a mutation-infinite quiver for which every subquiver is mutation-finite.

2.2. **Properties of minimal mutation-infinite quivers.** In their paper on the classification of mutation-finite quivers Felikson, Shapiro and Tumarkin prove a useful fact about minimal mutation-infinite quivers.

Theorem 2.7 ([4, Lemma 7.3]). *Any minimal mutation-infinite quiver contains at most 10 vertices. Equivalently, any mutation-infinite quiver of size greater than 10 must contain a mutation-infinite subquiver.*

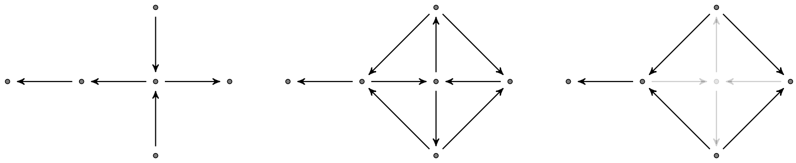


Figure 4. The left quiver is minimal mutation-infinite. Mutation at the central vertex yields the central quiver however this is not minimal mutation-infinite. Removing the central vertex gives the right quiver, which is also minimal mutation-infinite.

An important restriction of the minimal mutation-infinite property of quivers is that it is not preserved by mutation. An example of a mutation which does not preserve the minimal mutation-infinite property of a quiver is given in Figure 4. However there are some specific mutations which do preserve this property, of which sink-source mutations are an example.

Definition 2.8. A **sink** is a vertex in a quiver such that all adjacent arrows are directed into that vertex, whereas a **source** is a vertex such that all adjacent arrows are directed away from the vertex. Define a **sink-source mutation** as a mutation at either a sink or a source.

Proposition 2.9. *Sink-source mutations of a quiver preserve whether it is minimal mutation-infinite or not.*

Proof. A mutation at a sink (resp. source) reverses the direction of all arrows adjacent to it, so the vertex becomes a source (sink).

Let P be a minimal mutation-infinite quiver, and Q a quiver obtained from P by a sink-source mutation at a vertex v. The quiver Q is mutation-equivalent to P, so is mutation-infinite. The subquiver of Q obtained by removing the mutated vertex v is precisely the same as the subquiver of P constructed by removing v. Any other subquiver of Q is a single sink-source mutation away from the corresponding subquiver of P. Every subquiver of P is mutation-finite, so every subquiver of Q is also mutation-finite, hence Q is minimal mutation-infinite. □

Remark 2.10. Any two orientations of an unoriented graph without cycles are mutation equivalent through a series of sink-source mutations by a result of [1], therefore if one orientation is minimal mutation-infinite then all other orientations are too. However an unoriented graph with cycles could have different orientations such that one is minimal mutation-infinite and another is not. Moreover different orientations of the same graph may differ in whether they are mutation-finite, see Figure 5 for an example.

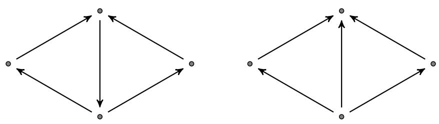


Figure 5. An example of different orientations of the same graph, the left quiver is mutation-finite, while the right quiver is minimal mutation-infinite.

The following well known fact limits the possible quivers which could be minimal mutation-infinite.

Proposition 2.11. *If Q is a mutation-finite quiver with at least 3 vertices then the number of arrows between any two vertices of Q is at most* 2.

A comprehensive proof of this fact can be found in Derksen and Owen's paper introducing a previously unknown mutation-finite quiver [3, Section 3]. This is equivalent to stating that any quiver with 3 or more arrows between any two vertices is necessarily mutation-infinite.

Every subquiver of a minimal mutation-infinite quiver is mutation-finite and so each subquiver has at most 2 arrows between any two vertices. Therefore the minimal mutation-infinite quiver itself has at most 2 arrows between any two vertices.

Proposition 2.12. *Any mutation-infinite quiver with 3 vertices is minimal mutation-infinite.*

Proof. All quivers with only 2 vertices are mutation-finite, as mutation at either vertex just reverses the direction of the arrows. Hence the mutation-class contains just these two quivers. □

3. Coxeter simplices

It is known that hyperbolic Coxeter simplices of finite volume exist up to dimension 9 and so admit diagrams with up to 10 vertices. In the following section we explore the links between these diagrams and the minimal mutation-infinite quivers which also exist with up to 10 vertices, as stated in Theorem 2.7.

An n-dimensional Coxeter simplex is considered in one of three spaces: spherical, Euclidean and hyperbolic. As a simplex they are the convex hull of $n+1$ points and so have $n+1$ facets.

Definition 3.1. A simplex is a **Coxeter simplex** if the hyperplanes which make up the faces have dihedral angles all submultiples of π. In the case of hyperbolic Coxeter simplices we allow the case where the planes meet at the boundary and so have dihedral angle 0.

Given a Coxeter simplex we denote the hyperplanes by H_i and the angle between hyperplanes H_i and H_j by $\frac{\pi}{k_{ij}}$.

Definition 3.2. The **Coxeter diagram** associated to a Coxeter simplex is an unoriented graph with a vertex i for each hyperplane H_i and a weighted edge between vertices i and j when $k_{ij} > 3$ with weight k_{ij}. We add an unweighted edge between i and j when $k_{ij} = 3$, and if the angle between two hyperplanes H_i and H_j is $\frac{\pi}{2}$ then no edge is put between i and j.

In the hyperbolic case, where two hyperplanes meet at the boundary, then the edge is given weight ∞.

The Coxeter group associated to a given Coxeter diagram is constructed from the following representation, where each generator s_i represents reflection in the hyperplane H_i,

$$\left\langle s_i \;\middle|\; s_i^2 = 1 = (s_i s_j)^{k_{ij}} \right\rangle.$$

3.1. **Simply-laced Coxeter simplex diagrams in different spaces.**

Definition 3.3. **Simply-laced** Coxeter diagrams are those where $k_{ij} \in \{2, 3\}$ for all i and j.

This is equivalent to only allowing angles of $\frac{\pi}{2}$ and $\frac{\pi}{3}$ in the Coxeter simplex. Simply-laced Coxeter diagrams only contain edges with no weights, and so a quiver can be constructed from the diagram by choosing an orientation for each edge.

Coxeter simplices can be considered over spherical, Euclidean or hyperbolic space. In each case the quivers obtained by choosing an orientation for the simply-laced Coxeter diagrams have different properties. The following are well known results about the spherical and Euclidean cases.

Remark 3.4. In [2], Coxeter classified simply-laced spherical Coxeter simplex diagrams as Dynkin diagrams of type A, D and E. Orientations of these diagrams are mutation-finite quivers and give finite-type cluster algebras, as shown in Fomin and Zelevinsky's classification of finite-type cluster algebras [6].

Similarly, simply-laced Euclidean Coxeter simplex diagrams are affine Dynkin diagrams of type $\tilde{A}$, $\tilde{D}$ and $\tilde{E}$. Felikson, Shapiro and Tumarkin's mutation-finite classification [4] shows that orientations of these diagrams are mutation-finite but give infinite-type cluster algebras.

It is known that the hyperbolic Coxeter simplex diagrams satisfy the following property.

Remark 3.5. Any subdiagram of a simply-laced hyperbolic Coxeter simplex diagram is either a Dynkin or an affine Dynkin diagram.

This follows from Theorems 3.1 and 3.2 of Vinberg's paper [11] concerning the reflection groups generated by the reflections in n hyperplanes of an n dimensional hyperbolic Coxeter simplex.

3.2. **A family of minimal mutation-infinite quivers.** Given a simply-laced hyperbolic Coxeter simplex diagram, construct a quiver by choosing an orientation on each edge. From Remark 3.5, any subquiver of this quiver will be an orientation of either a Dynkin diagram or an affine Dynkin diagram and so Remark 3.4 shows that any subquiver is mutation-finite.

Using the classification of mutation-finite quivers given in [4], it can be seen that almost all orientations of hyperbolic Coxeter simplex diagrams are mutation-infinite. There is precisely one mutation-finite orientation of hyperbolic Coxeter simplex diagrams for each size k between 5 and 9, with two of size 4, as shown in Figure 6.

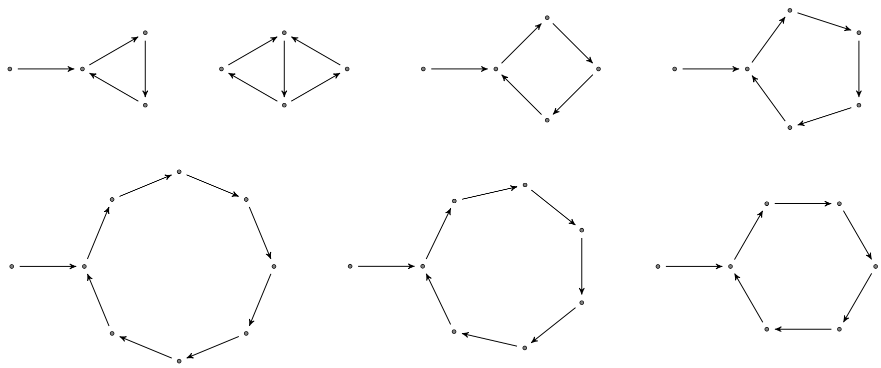


FIGURE 6. Mutation-finite orientations of hyperbolic Coxeter simplex diagrams

It follows from Remarks 3.4 and 3.5 that all mutation-infinite orientations of hyperbolic Coxeter simplex diagrams are in fact minimal mutation-infinite quivers. This then raises the question of whether all minimal mutation-infinite quivers can be given in this form or not.

Proposition 3.6. *There exist minimal mutation-infinite quivers which are not orientations of a hyperbolic Coxeter simplex diagram for all sizes of quiver from 5 to 10.*

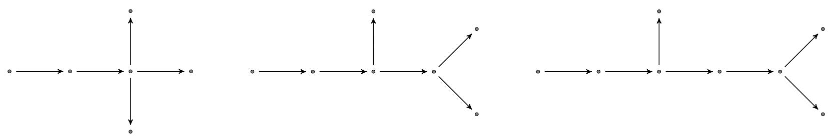


FIGURE 7. Orientations of tree-like hyperbolic Coxeter simplex diagrams of size 6, 7 and 8.

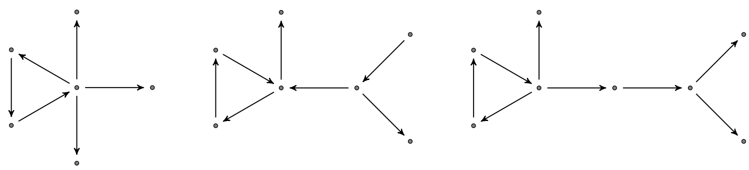


FIGURE 8. Minimal mutation-infinite quivers not orientations of hyperbolic Coxeter simplex diagrams.

Proof. To prove this it suffices to give an example of such a quiver for each size. The construction of this quiver for size $6 \leq k \leq 10$ is as follows:

Take the tree-like hyperbolic Coxeter simplex diagram of size k and orient it in such a way that all arrows point the same way as illustrated in Figure 7. This quiver Q is minimal mutation-infinite as shown above, and contains an orientation of the Dynkin diagram A_3 as a subquiver. Mutating at the centre vertex of this A_3 creates an oriented triangle in the resulting quiver P, giving the quivers in Figure 8 which are not orientations of hyperbolic Coxeter simplex diagrams.

The resulting quiver is mutation-equivalent to the orientation of a hyperbolic Coxeter simplex, so is mutation-infinite. Each subquiver obtained by removing vertex n from P is either the same as the subquiver obtained by removing n from Q, or a single mutation away from it. Hence as Q is minimal mutation-infinite, all such subquivers are mutation-finite and so P is also minimal mutation-infinite.

The only minimal mutation-infinite quivers with 5 vertices are of the form shown in Figure 9. Mutation of an orientation of a hyperbolic Coxeter simplex diagram gives such a quiver, and all subquivers are mutation-equivalent to subquivers of the initial quiver so the resulting quiver is again minimal mutation-infinite. □

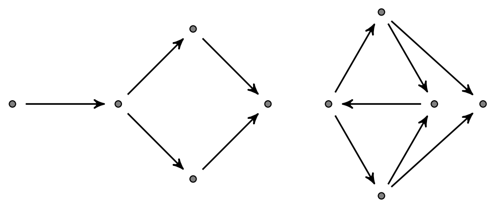


FIGURE 9. The quiver on the left is an orientation of a hyperbolic Coxeter simplex diagram. Mutation at the trivalent vertex yields the quiver on the right which is also minimal mutation-infinite.

4. Minimal mutation-infinite quiver moves

Orientations of hyperbolic Coxeter simplex diagrams give a family of minimal mutation-infinite quivers, however Proposition 3.6 shows the existence of other minimal mutation-infinite quivers. This section discusses the approach taken to classify all such quivers.

Many examples of minimal mutation-infinite quivers are only a small number of mutations away from an orientation of a hyperbolic Coxeter diagram. As discussed in Section 2.2 mutations do not in general preserve the minimal mutation-infinite property of a quiver, however it can be proved that specific mutations, where a vertex is surrounded by a particular subquiver, do indeed preserve this property. An example of such a mutation was used in the proof of Proposition 3.6. These particular mutations which preserve the minimal mutation-infinite property can be considered as **moves** among all minimal mutation-infinite quivers.

As mutation acts by changing the quiver locally around the mutated vertex, while leaving arrows further from the vertex fixed, these moves can be defined in terms of the subquivers which change under the mutations. In this way applying the move is equivalent to replacing some subquiver with a different subquiver.

The minimal mutation-infinite preserving mutations often depend on some restriction of how the vertices in the subquivers are connected in the whole quiver outside the subquiver. This data then needs to be encoded in the moves along with the subquivers.

Definition 4.1. When referred to in a move, a **line** is a line of vertices such that one end point is connected to the move subquiver. A line of length zero consisting of just a single vertex is also considered valid.

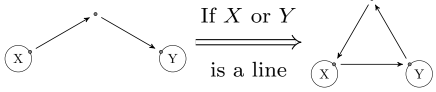


FIGURE 10. An example of a minimal mutation-infinite move.

4.1. A move example.

Figure 10 gives an example of one such move. The move is applied to a quiver by mutating at the central vertex. The circles labelled X and Y denote connected components of the quiver fixed by the move. The vertex on the boundary of X is considered to be contained in X. In this case the move requires that one of the components be a line (or just

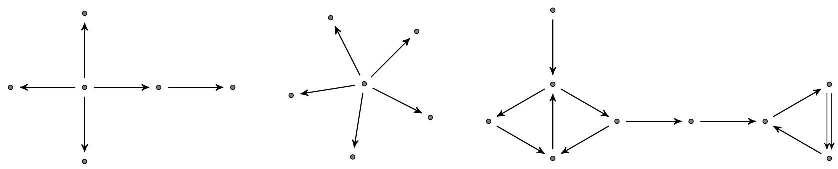


FIGURE 11. The move in Figure 10 applies to the first quiver, but does not apply to the others. The second quiver does not contain either subquiver, while the third does, but neither component is a line with an endpoint in the subquiver.

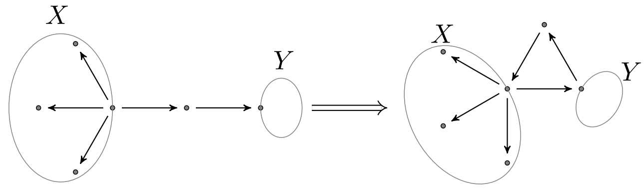


FIGURE 12. An example of how the move in Figure 10 changes a quiver. Note that Y consists of a single vertex, and so can be thought of as a line of length zero.

the single vertex) for the move to apply. Figure 11 shows some examples of quivers for which this move is applicable or not and Figure 12 shows how it acts on the first quiver in Figure 11.

Proposition 4.2. *The image of a minimal mutation-infinite quiver under the move in Figure 10 is minimal mutation-infinite.*

Proof. Let P be the initial quiver and Q its image under the move. Denote the vertex at which mutation occurs during the move as w.

The move is equivalent to mutation at w hence Q is in the same mutation class as P. P is mutation-infinite so Q is also mutation-infinite.

Any subquiver Q' of Q, obtained by removing a vertex v in either X or Y, will contain w. Mutating at w will yield a quiver $\mu(Q')$ that is equal to one obtained by removing the corresponding vertex v from P. As P is minimal mutation-infinite, such a subquiver of P is necessarily mutation-finite, hence $\mu(Q')$ is mutation-finite and so Q' is also mutation-finite.

Removing w gives a subquiver Q' of Q which is not mutation-equivalent to a subquiver of P. Instead the extra condition that either X or Y is a line ensures that this quiver is a subquiver of P by removing the vertex at the end of that line, and so is mutation-finite. For example consider the quivers in Figure 12, removing w from Q gives a quiver which is the same as one obtained by removing the vertex in Y from P.

Hence Q is minimal mutation-infinite. □

The proofs for all moves are similar to this. The moves are always constructed from sequences of mutations, so the image is mutation-infinite and the quivers obtained by removing vertices outside those vertices which are mutated by the move can always be mutated back to a subquiver of the initial quiver. The challenge is determining whether a quiver obtained by removing a vertex at which one of the mutations took place is mutation-equivalent to a subquiver of the initial quiver.

Proposition 4.3. *The move given by reversing the move in Figure 10 is a valid move.*

Proof. As discussed above, it suffices to show that removing the vertex at which the mutation occurs yields a mutation-finite quiver. Denote the initial quiver as P, the image Q and the mutated vertex w.

Removing w from Q gives a quiver R which is the disjoint union of X and Y, therefore R is mutation-finite if and only if both X and Y are.

Both X and Y are contained in P, so are subquivers of P and hence are mutation-finite. Therefore R is also mutation-finite, so Q is minimal mutation-infinite. □

Appendix B contains a list of all moves necessary to classify minimal mutation-infinite quivers.

The moves required to classify all minimal mutation-infinite quivers up to size 9 only have requirements that certain components are lines or are connected to other components by lines. For the size 10 quivers, stricter conditions are required as some moves require that a certain quiver constructed from the components is mutation-finite. These moves are of the form

$$\left\{\begin{array}{c} A \text{ is minimal mutation-infinite} \\ \text{and} \\ B \text{ is mutation-finite} \end{array}\right\} \iff \left\{\begin{array}{c} C \text{ is minimal mutation-infinite} \\ \text{and} \\ D \text{ is mutation-finite} \end{array}\right\},$$

where A and C are rank 10 quivers, B is a subquiver of C and D is a subquiver of A. If A is minimal mutation-infinite then as a subquiver D is mutation-finite, however this is not sufficient to show that C is minimal mutation-infinite, as there is no way to determine from A whether B, a subquiver of C, is mutation-finite. If B is given to be mutation-finite then the move applies and so C is minimal mutation-infinite. On the other hand if C is minimal mutation-infinite then as a subquiver B is mutation-finite, but D cannot be shown to be mutation-finite just by considering C. The B and D quivers constructed in such a way are always of a smaller size and so the results for smaller size quivers can be applied. These moves are still involutions as, after applying the move once, the conditions are automatically satisfied to apply the same move in reverse.

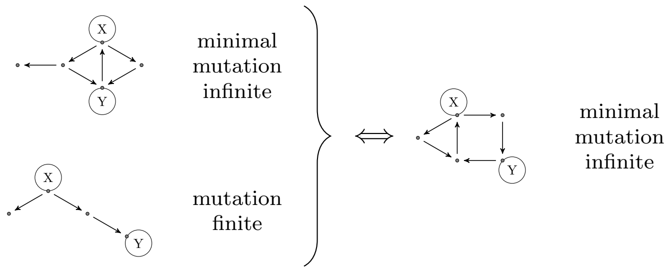


FIGURE 13. Example of a size 10 move with added constraints

Figure 13 gives an example of one such move for size 10 quivers. In one direction the move applies without any additional constraints, but in the other direction the move requires that a certain quiver constructed from quiver components is mutation-finite.

5. Classifying minimal mutation-infinite quivers

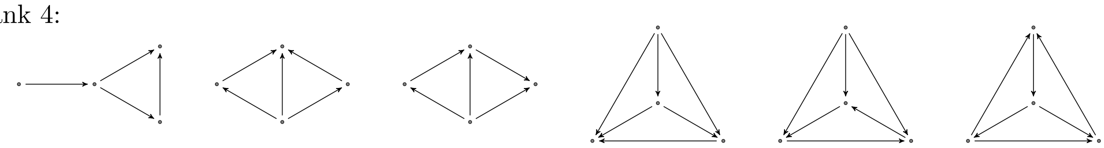
Define an equivalence relation where two quivers are equivalent if one quiver can be obtained from the other through a sequence of moves. Then these moves together with the list of representatives (see Tables 1, 2 and 3) classify minimal mutation-infinite quivers.

Hyperbolic Coxeter simplex diagrams give a family of minimal mutation-infinite quivers, and so orientations of these diagrams are some of the representatives of the classes. Caldero and Keller proved that any two acyclic orientations of a diagram, belonging to the same mutation class, are mutation-equivalent through a sequence of sink-source mutations in [1, Corollary 4]. As a result of this, given any hyperbolic Coxeter simplex diagram the classification requires a representative for each acyclic orientation which can not be obtained from any other acyclic orientation by sink-source mutations.

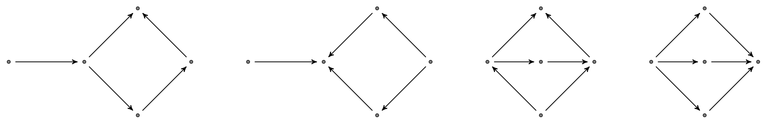
Many minimal mutation-infinite quivers can be transformed into one of the hyperbolic Coxeter diagrams, however there are some which can not. Therefore the classification contains hyperbolic Coxeter classes and some exceptional classes. A particular case of these exceptional cases arises from those minimal mutation-infinite quivers which contain a double arrow between two vertices. There are two such classes for quivers of size 6 and one class for each size between 7 and 10.

The result places a bound on the number of moves required to transform any minimal mutation-infinite quiver to one of the class representatives. Diagrams of the representatives can be found in Tables 1, 2 and 3. This statement can then be reversed to give a construction of all possible minimal mutation-infinite quivers from these representatives. The procedure to do this would be progressively applying the moves to the set of all quivers computed so far. As the number of moves is bounded this procedure will stop and at that point all minimal mutation-infinite quivers will have been computed.

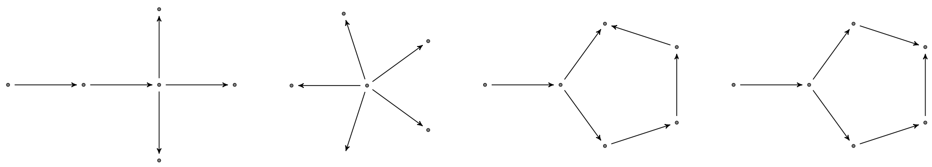
Rank 4:



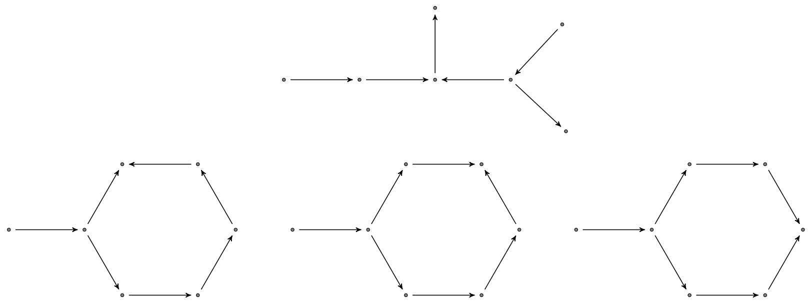
Rank 5:



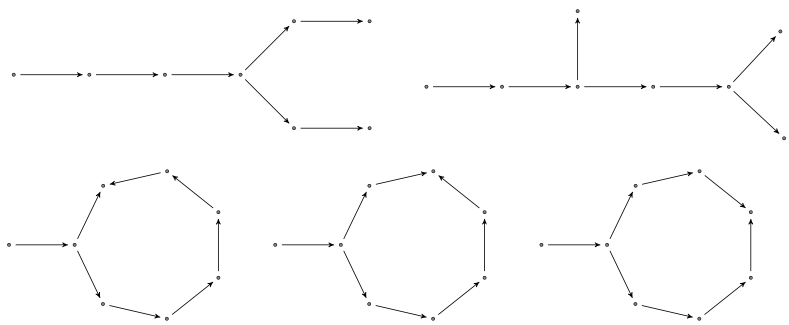
Rank 6:



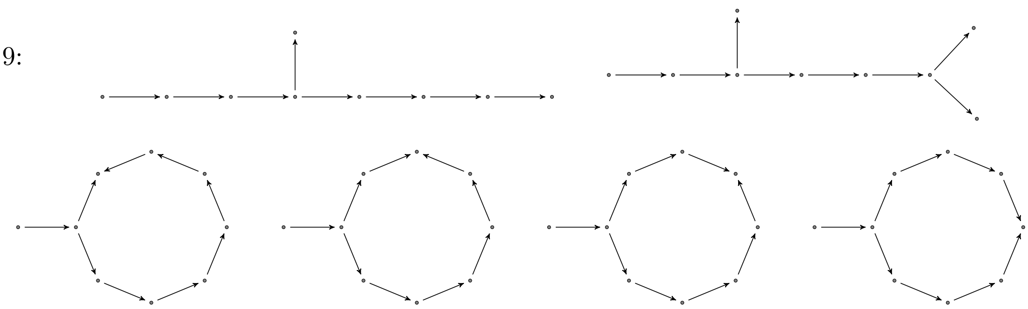
Rank 7:



Rank 8:



Rank 9:



Rank 10:

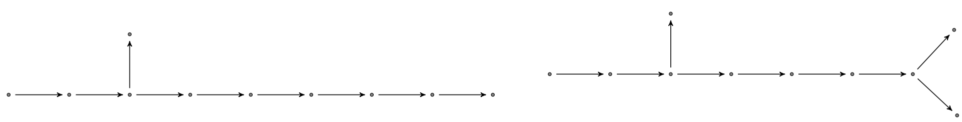


Table 1. Representatives: Orientations of hyperbolic Coxeter simplex diagrams

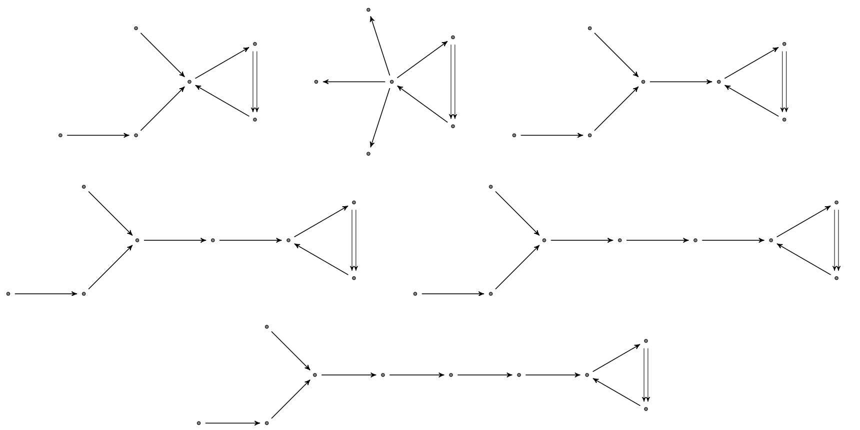


TABLE 2. Representatives: Double arrow quivers

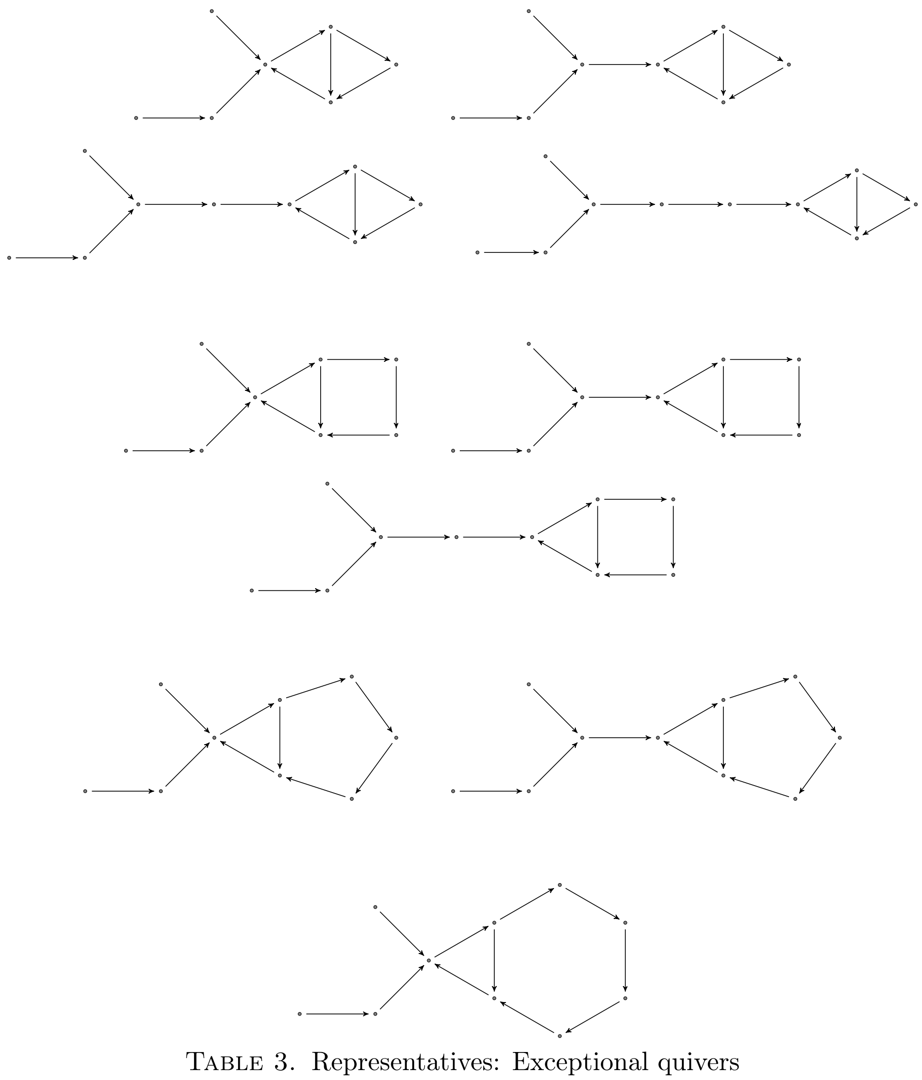


TABLE 3. Representatives: Exceptional quivers

Theorem 5.1. *Any minimal mutation-infinite quiver with at most* 9 *vertices can be transformed through sink-source mutations and at most 5 moves to one of an orientation of a hyperbolic Coxeter diagram, a double arrow quiver or an exceptional quiver (see Tables 1–3).*

As discussed in Section 4 above the moves required for quivers of size 10 have more constraints and are more complicated than those for smaller quivers. As such this result needs to be restated when considering these larger quivers.

Theorem 5.2. *Any minimal mutation-infinite quiver can be transformed through sink-source mutations and at most 10 moves to one of an orientation of a hyperbolic Coxeter diagram, a double arrow quiver or an exceptional quiver (see Tables 1–3).*

Appendix A discusses the computations used to verify this result and find all minimal mutation-infinite quivers. There are in total 18,799 such quivers (excluding those with 3 vertices) which are orientations of 574 different graphs. Pictures of all minimal mutation-infinite quivers organised into their move-classes can be found on the author's website:
`http://www.math.dur.ac.uk/users/j.w.lawson/mmi/quivers`.

5.1. **A mutation-infinite check using minimal mutation-infinite quivers.** Any mutation-infinite quiver must contain some minimal mutation-infinite quiver as a subquiver. Hence given a list of all the minimal mutation-infinite quivers there is an algorithm to check whether a given quiver is mutation-infinite without having to compute any part of its mutation class.

Let Q be a possibly mutation-infinite quiver and $\{P_i\}_{i\in I}$ be all minimal mutation-infinite quivers indexed by I. For each $i \in I$ if P_i is a subquiver of Q then Q is mutation-infinite, otherwise continue to the next i. If no minimal mutation-infinite quiver is in fact a subquiver of Q then Q is mutation-finite.

References

[1] Philippe Caldero and Bernhard Keller. From triangulated categories to cluster algebras. II. *Ann. Sci. École Norm. Sup. (4)*, 39(6):983–1009, 2006.

[2] H. S. M. Coxeter. Discrete groups generated by reflections. *Ann. of Math. (2)*, 35(3):588–621, 1934.

[3] Harm Derksen and Theodore Owen. New graphs of finite mutation type. *Electron. J. Combin.*, 15(1):Research Paper 139, 15, 2008.

[4] Anna Felikson, Michael Shapiro, and Pavel Tumarkin. Skew-symmetric cluster algebras of finite mutation type. *J. Eur. Math. Soc. (JEMS)*, 14(4):1135–1180, 2012.

[5] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[6] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.*, 154(1):63–121, 2003.

[7] Bernhard Keller. Categorification of acyclic cluster algebras: an introduction. In *Higher structures in geometry and physics*, volume 287 of *Progr. Math.*, pages 227–241. Birkhäuser/Springer, New York, 2011.

[8] John Lawson. *Minimal mutation-infinite quivers*, 2015. `http://www.math.dur.ac.uk/users/j.w.lawson/mmi`.

[9] Ahmet I. Seven. Recognizing cluster algebras of finite type. *Electron. J. Combin.*, 14(1):Research Paper 3, 35 pp. (electronic), 2007.

[10] Michael Shapiro. *Quivers of finite mutation type*, 2010. `http://www.math.msu.edu/~mshapiro/FiniteMutation.html`.

[11] È. B. Vinberg. Hyperbolic groups of reflections (Russian). *Uspekhi Mat. Nauk*, 40(1):29–66, 255, 1985. English translation in: *Russ. Math. Surv.*, 40(1):31–75, 1985.

[12] Lauren K. Williams. Cluster algebras: an introduction. *Bull. Amer. Math. Soc. (N.S.)*, 51(1):1–26, 2014.

Dept. of Mathematical Sciences, Durham University, South Road, Durham, DH1 3LE, UK.

Email: `j.w.lawson@durham.ac.uk`

Appendix A. Computing minimal mutation-infinite quivers

The quiver classification involved a large computational effort to find all minimal mutation-infinite quivers. This section details the procedures used in this computation. Details about implementations of these procedures can be found on the author's website [8].

A.1. **Finding the size of a mutation-class.** An important computation in all the following algorithms is determining whether a given quiver is mutation-finite or mutation-infinite. There is a fast approximation which can prove a quiver is mutation-infinite and a slower procedure which proves a quiver is mutation-finite.

A.1.1. *Fast approximation to check whether a quiver is mutation-infinite.* Proposition 2.11 states that any mutation-finite quiver has at most 2 arrows between any two vertices. This gives a procedure that can prove that a quiver is mutation-infinite, but which cannot prove that a quiver is mutation-finite. This procedure was used in computations by Felikson, Shapiro and Tumarkin in their classification of skew-symmetric mutation-finite quivers [4] and Shapiro's comments on the procedure can be found on his website [10].

The procedure, given in Algorithm A.1, checks whether the quiver contains 3 or more arrows between any two vertices, if it does then the quiver is mutation-infinite. Otherwise, pick a vertex at random and mutate the quiver at this vertex and repeat with this new quiver.

Input: Q Quiver to check
Data: M Number of mutations to perform
Data: k Counter initially 0
Result: Whether Q is mutation-infinite, or probably mutation-finite

while $k < M$ **do**
    if *Q contains 3 or more arrows between 2 vertices* **then**
        return *Q is mutation-infinite*
    Choose a random vertex
    Mutate Q at this vertex
    Increment k
return *Q is probably mutation-finite*

Algorithm A.1: Fast approximation whether a quiver is mutation-infinite

For mutation-finite quivers this process would never terminate without the bound on the number of mutations, and it is possible that for mutation-infinite quivers the randomly chosen mutations never generate an edge with more than 2 arrows. Therefore this is only an approximation and a maximum number of mutations should be attempted before stopping. If no quiver was found with more than 2 arrows between two vertices then, provided the number of mutations was high, the quiver is probably mutation-finite.

A.1.2. *Computing a full finite mutation-class.* While the above procedure can show a quiver is probably mutation-finite, we require a procedure that can definitively prove it. To do this the whole mutation-class of the quiver must be found.

The algorithm to find the mutation-class of a mutation-finite quiver calculates the whole exchange graph of the initial quiver. First compute all mutations of this quiver, then for each of these quivers compute all mutations and continue until no further quivers are computed. By keeping track of which mutations link two vertices, only those mutations which are not known need to be computed. See Algorithm A.2.

A.1.3. *Slower mutation-finite check.* The above algorithm will only terminate if the initial quiver is mutation-finite. In the case of a mutation-infinite quiver, the mutation-class is infinite, so the computation will continue indefinitely. The algorithm can be adapted to terminate for mutation-infinite quivers using the result in Proposition 2.11.

Once a new quiver is computed which has not yet been found, check whether it contains three or more arrows between any two vertices. If it does then the mutation-class is known to be infinite, so the procedure can be terminated. See Algorithm A.3.

Input: Q Quiver
Data: L Queue of quivers to mutate
Data: A List of all quivers found in the mutation class so far
Data: M_P For each quiver P, a map taking a vertex in P to the quiver obtained by mutating P at that vertex (if the mutation has been computed)
Result: A List of all quivers in the mutation class

```
Add Q to L
while L is not empty do
    Remove quiver P from the top of queue L
    for i = 1 to (Number of vertices) do
        if M_P has a quiver at vertex i then
            Continue to next vertex
        else
            Let P' be the mutation of P at i
            if P' ∈ A then
                Update M_P' so vertex i points to P
            else
                Create M_P' with vertex i pointing to P
                Add P' to A
                Add P' to L
            Update M_P so vertex i points to P'
return A
```

Algorithm A.2: Compute mutation-class of a mutation-finite quiver

Input: Q Quiver
Data: L Queue of quivers to mutate
Data: A List of all quivers found in the mutation class so far
Data: M_P For each quiver P, a map taking a vertex in P to the quiver obtained by mutating P at that vertex (if the mutation has been computed)
Result: Whether Q is mutation-infinite or not

```
Add Q to L
while L is not empty do
    Remove quiver P from the top of queue L
    for i = 1 to (Number of vertices) do
        if M_P has a quiver at vertex i then
            Continue to next vertex
        else
            Let P' be the mutation of P at i
            if P' ∈ A then
                Update M_P' so vertex i points to P
            else
                if P' has more than 3 arrows between 2 vertices then
                    return Q is mutation-infinite
                Create M_P' with vertex i pointing to P
                Add P' to A
                Add P' to L
            Update M_P so vertex i points to P'
return Q is mutation-finite
```

Algorithm A.3: Determine whether a quiver is mutation-infinite or not

There are only a finite number of ways to draw a graphs with a fixed number of vertices and up to 2 arrows between any two vertices. Hence in an infinite mutation-class there will eventually be a quiver with 3 or more arrows between two vertices and therefore the procedure will always terminate.

The two procedures to compute whether a quiver is mutation-finite can be combined to provide a faster run time in the majority of cases. By first using the fast approximation, most mutation-infinite quivers will be identified as mutation-infinite and any quivers which are not then get passed to the slower check to confirm whether they are mutation-finite.

A.2. **Computing quivers.** The above algorithms give procedures to tell whether a given quiver is mutation-finite or mutation-infinite. By iterating through a range of quivers these checks can be used to find all quivers which satisfy certain properties.

A.2.1. *Computing all mutation-finite quivers.* Proposition 2.5 states that all subquivers of a mutation-finite quiver are again mutation-finite. This fact is used to build up mutation-finite quivers of a certain size n by adding vertices to the mutation-finite quivers with $n-1$ vertices. All 2 vertex quivers are mutation-finite, so with these as a starting point we can recursively compute all mutation-finite quivers of size n, using the procedure in Algorithm A.4.

By Proposition 2.11 any mutation-finite quiver contains at most 2 arrows between any two vertices, so when adding a vertex to the quivers of size $n-1$ it suffices to only add either 0, 1 or 2 between the new vertex and any others. Adding more arrows would immediately yield a mutation-infinite quiver.

Input: n Size of quiver to output
Data: A List of mutation-finite quivers
Result: A list of all mutation-finite quivers of size n

Function `Finite(`*size* n`)`
    if $n = 2$ **then**
        Let $A = \{\cdot \to \cdot, \cdot \rightrightarrows \cdot\}$
        return A
    foreach *Quiver* Q *in* `Finite(`$n-1$`)` **do**
        foreach *Extension of* Q *to possibly mutation-finite quiver* Q' **do**
            if Q' *is mutation-finite* **then**
                Add Q' to A
    return A

Algorithm A.4: Compute all mutation-finite quivers of size n

A.2.2. *Computing all minimal mutation-infinite quivers.* Any subquiver of a minimal mutation-infinite quiver is a mutation-finite quiver. Therefore to construct these quivers of a certain size n start with all mutation-finite quivers of size $n-1$ and extend the quiver by adding another vertex in all possible ways with either 0, 1 or 2 arrows between the new vertex and any other vertices. The quivers obtained in this way could then be minimal mutation-infinite and so this needs to be verified.

For a given quiver to be minimal mutation-infinite it must satisfy two conditions, namely that it is mutation-infinite and that every subquiver is mutation-finite. Both of these conditions can be checked using the above procedures.

A.3. **Checking number of moves.** Theorem 5.2 states the maximum number of moves required to transform any minimal mutation-infinite quiver to one of the class representatives. To compute this number each minimal mutation-infinite quiver is checked in turn to find the minimal number of moves needed to transform that quiver to its class representative.

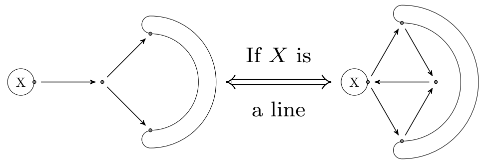
This minimal number of moves can be found by applying all applicable moves to the initial quiver and storing the number of moves taken to reach each quiver obtained in this way. We can ensure that the number of moves used to obtain a class representative is minimal by always choosing the next quiver used in the process to be the one obtained through the fewest number of moves.

Appendix B. List of moves

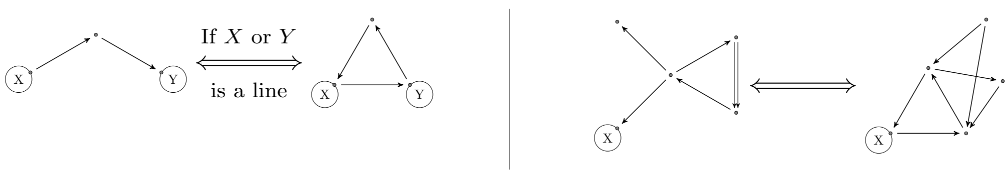
This section lists all moves required to transform any minimal mutation-infinite quiver to one of the representatives. Any listed move should also be considered along with the move where all arrows are reversed.

Where a move has the requirement that one of the components is a line this requires that the component is a line with one of its endpoints adjacent to the move subquiver.

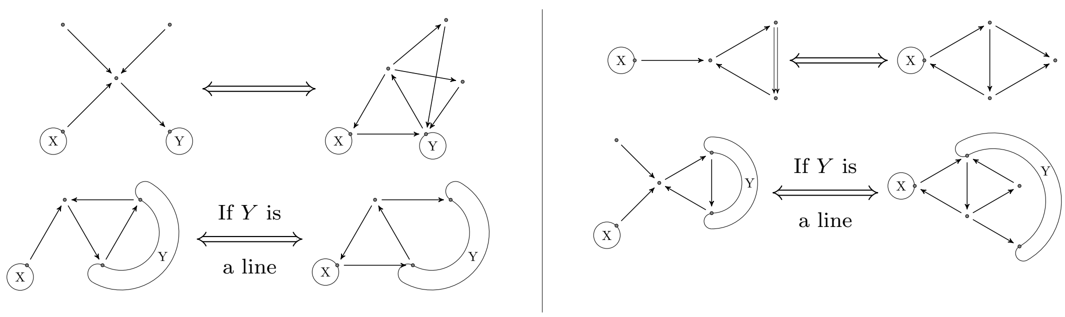
B.1. **Moves for quivers of size 5.**



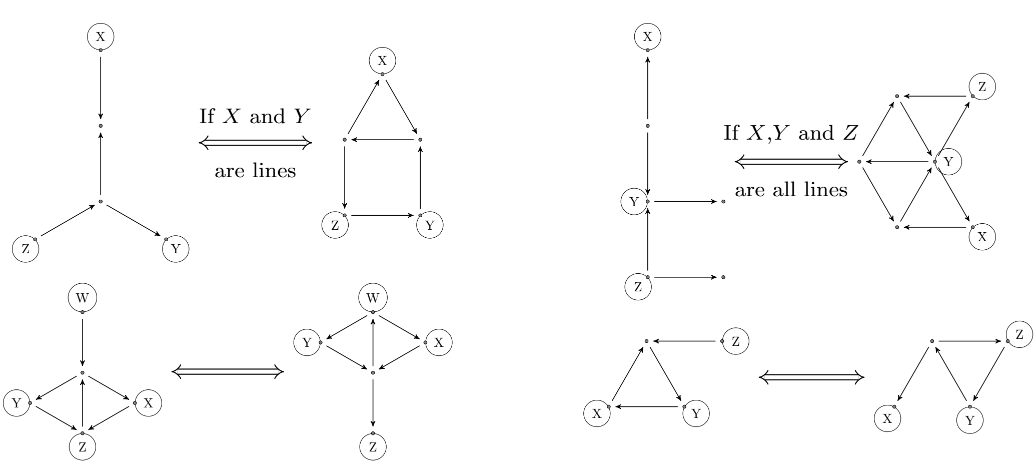
B.2. **Additional moves for quivers of size 6.**



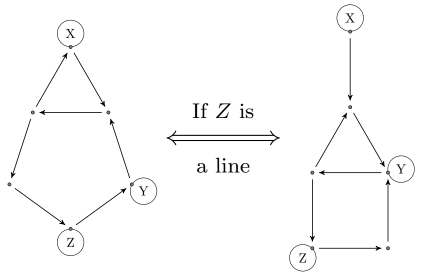
B.3. **Additional moves for quivers of size 7.**

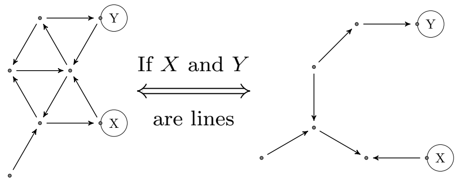


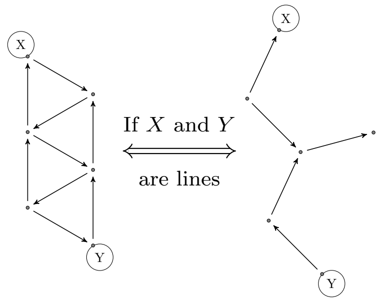
B.4. **Additional moves for quivers of size 8.**

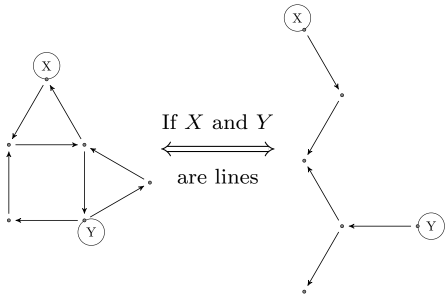


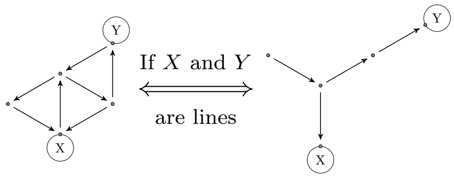
B.5. **Additional moves for quivers of size 9.**

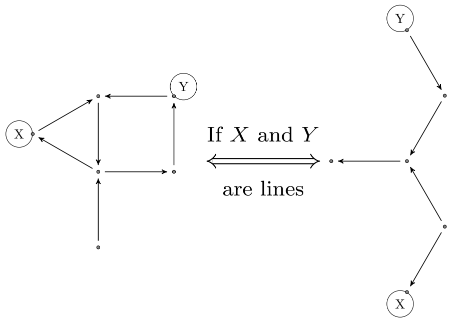


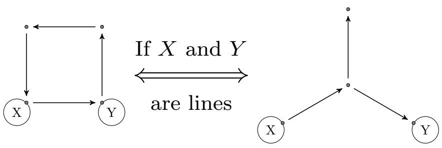


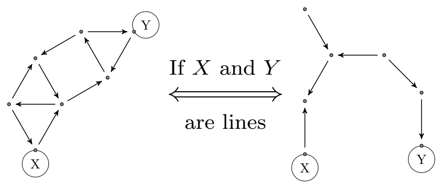


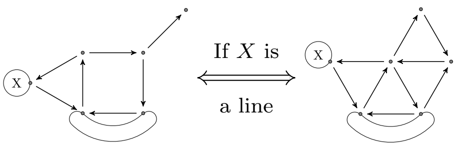


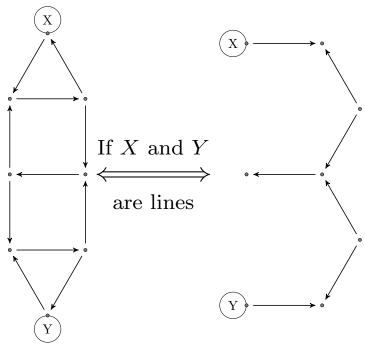


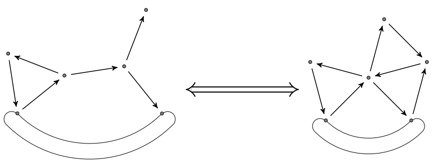


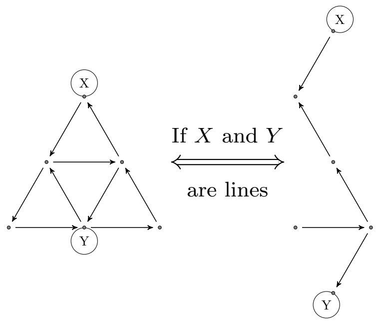




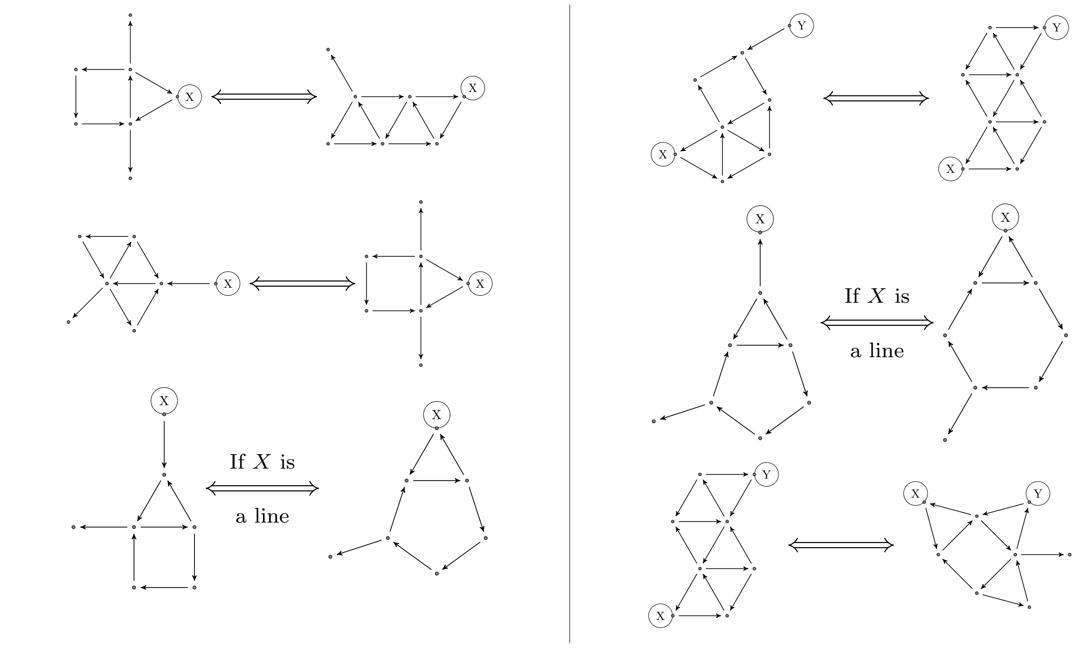


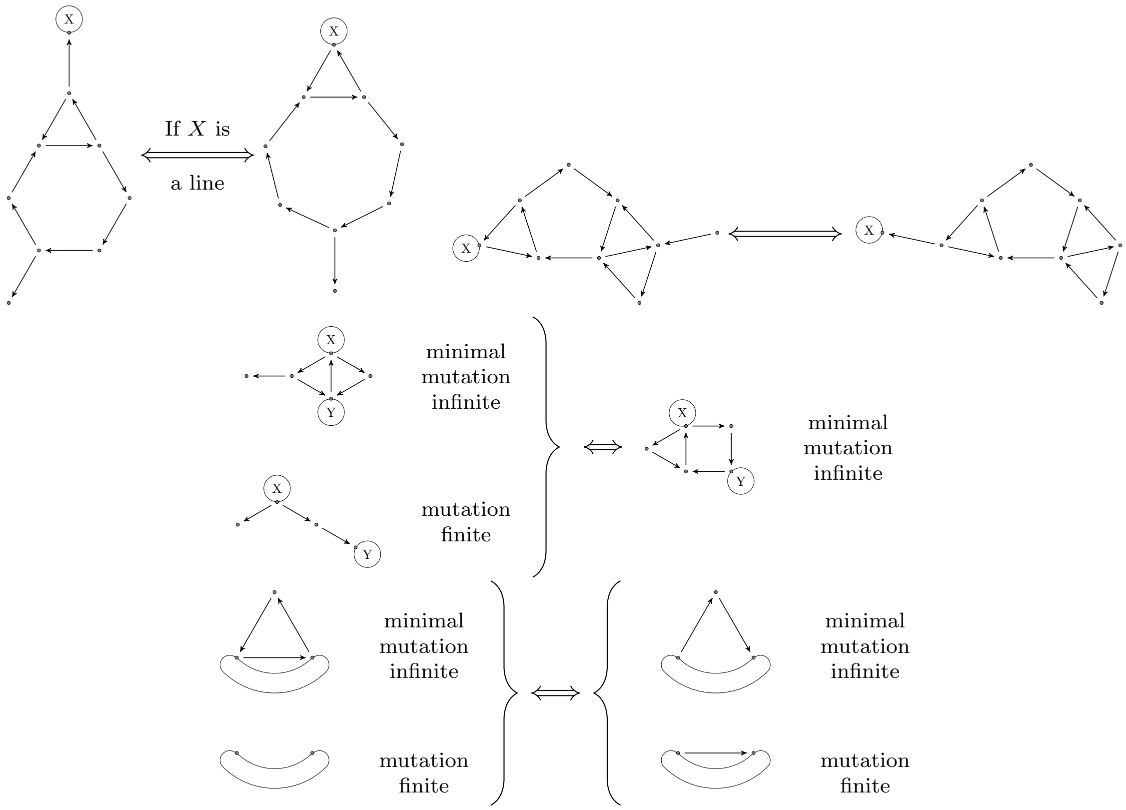


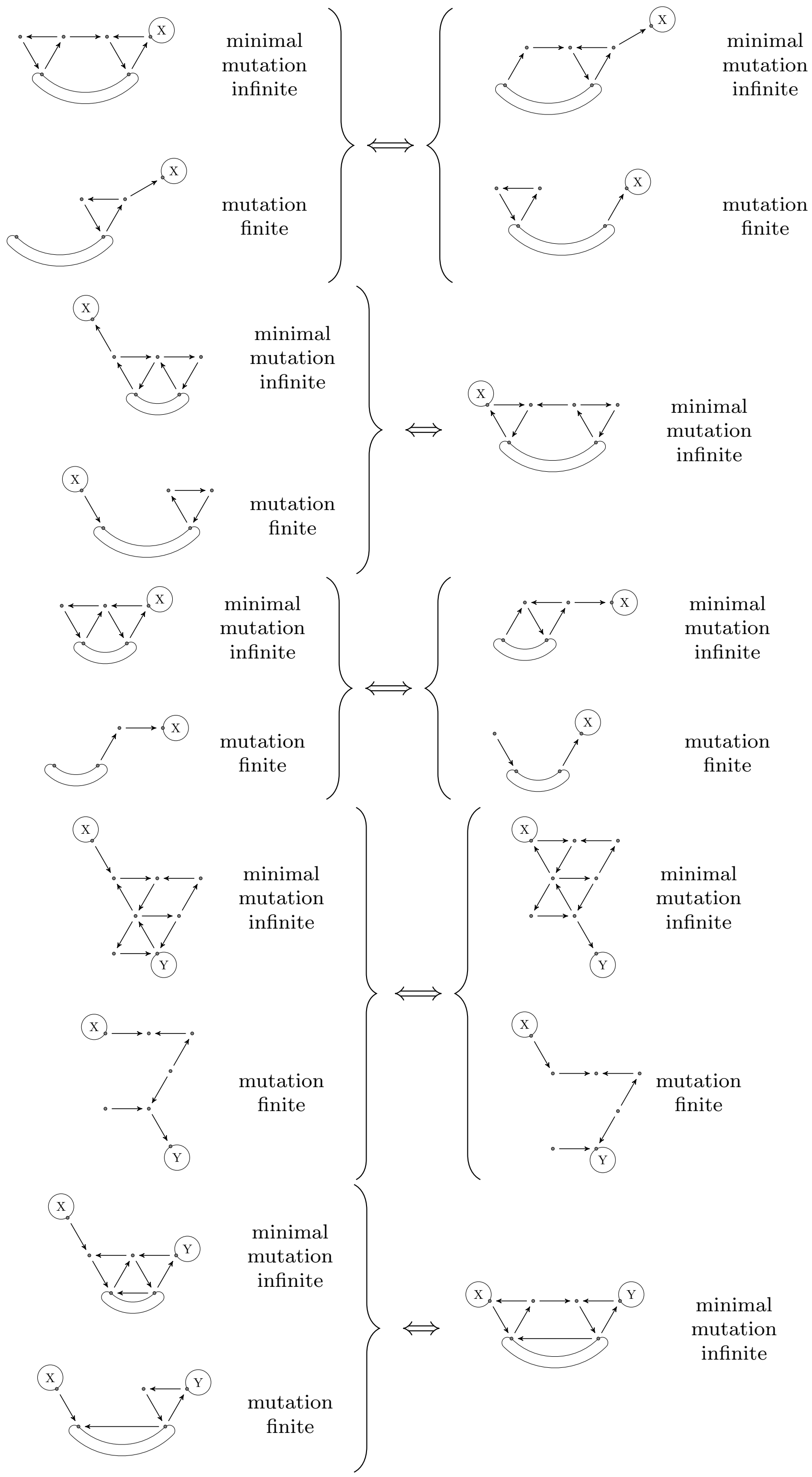


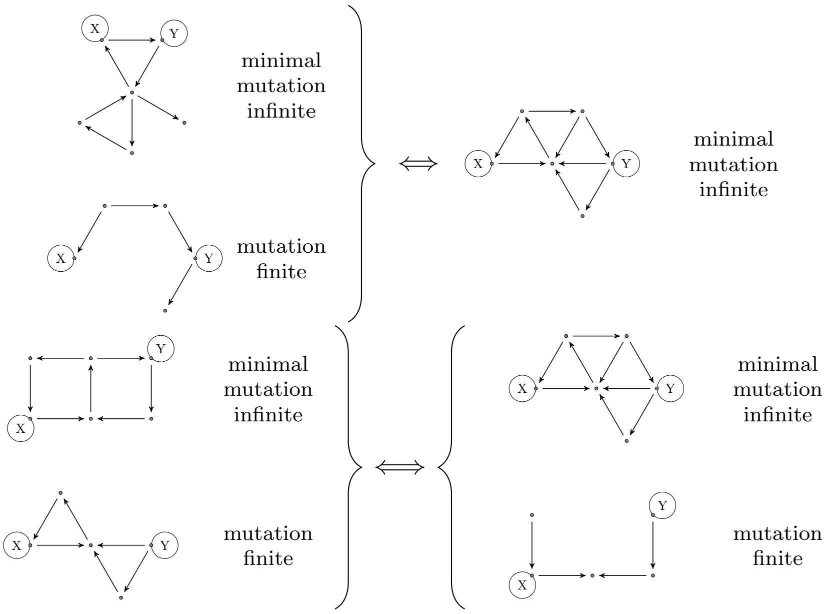


B.6. **Additional moves for quivers of size 10.**






X
minimal
mutation
infinite
X
minimal
mutation
infinite
X
mutation
finite
X
mutation
finite
X
minimal
mutation
infinite
X
minimal
mutation
infinite
X
mutation
finite
X
minimal
mutation
infinite
X
minimal
mutation
infinite
X
mutation
finite
X
mutation
finite
X
Y
minimal
mutation
infinite
X
Y
minimal
mutation
infinite
X
Y
mutation
finite
X
Y
mutation
finite
X
Y
minimal
mutation
infinite
X
Y
minimal
mutation
infinite
X
Y
mutation
finite


X
Y
minimal
mutation
infinite
X
Y
mutation
finite
X
Y
minimal
mutation
infinite
Y
X
minimal
mutation
infinite
X
Y
mutation
finite
X
Y
minimal
mutation
infinite
Y
X
mutation
finite